VERSION: Geophysical Fluid Dynamics-problems.doc
DATE: 2009-04-21

Geophysical Fluid Dynamics: Problems

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1. Introduction

The present collections of problems serve as exercises during the 5 weeks course in Ocean Models II at Earth Sciences: Oceanography, Göteborg University. The book we use is "Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects" by Benoit Cushman-Roisin and Jean-Maria Beckers (2008). This is an extension of Cushman-Roisin (1994) but now also with numerical modelling included. The book will be published 2007 but I have been kindly allowed to use a draft of the book from Cushman-Roisin and from Academic Press. The problems are partly taken from this book together with solutions made available from Benoit Cushman-Roisin. Some problems are also constructed by my self. The compendium is divided into two parts. In the first part problems with solutions are given and in the second part problems without solutions are given. My inspiration is from Lars Oredsson who introduced me to Dynamical Meterology by using an excellent book (James R. Holton: An Introduction to Dynamic Meteorolgy), and pedagogic compendiums for solving different problems. For an introduction to numerical models the compendium by Omstedt (2007) is also used.

Week	Date	Time	Lecture	Exercises
13	26/3, 27/3	915-1100	Chapters 1-4, 7, Numerical aspects	B 4.1, B7.1, B7.3, B8.1
14	31/3, 1/4, 2/4	915-1100	Chapters 8-9, 20	B 9.1, B9.2, B8old-1, B8old-2
15	7/4, 8/4	915-1100	Chapters 20 papers from Stigebrandt	
16	14/4, 15/4, 16/4	915-1100	Chapters 11,13,14, 15	B 11.1,B13.1, B13.2
17	21/4		Summary	

Examination: 24/4 kl 900-1400

Included Chapters: 1.1-1.10, 1.11, 2.1-2.5, 3.1-3.7, 4.1-4.6, 7.1-7.4, 8.1-8.8, 9.1-9.6, 11.1-11.2, (11.3), 11.5-11.6, 13.1-13.3, Appendix B, 14.1 (14.2), (15.1-15.4), 20.1-20.6.

2. Problems with solution

Chapter 1.

Problem A 1. 1. Name three naturally occurring flows in the atmosphere (Cushman-Roisin and Beckers: Analytical Problems 1-1).

Answer: E.g. the Jet Stream, the Trade Winds, The Westerlies, the sea- breeze, hurricanes and typhoons.

Problem A 1.2. The sea breeze is a light wind blowing from the sea as the result of a temperature difference between land and sea. As this temperature diffrerence reverses from day to night, the daytime sea breeze turns into a nighttime land breeze. If you were to construct a numerical model of the sea-land breeze, should you include the effect of the planetary rotation (Cushman-Roisin and Beckers: Physical problems 1-3)?

Answer: To evaluate the importance of the earth’s rotation while knowing the time scale, we should estimate the time ratio:

$$\omega = \frac{2\pi}{\Omega T} = \frac{24 hours}{T}$$

Where T is the sea-breeze time scale. The latter can be taken as 12 hours, because of the twice daily reversal in winds. This value yields ω=2, which is not much less than unity. Therefore, it is prudent to include the effects of the eatrh’s rotation in a sea-breeze model- [Actually, the rotational effects are far from being dominany, but to arrive at this conclusion requires an in-depth analysis].

Chapter 2.

Problem A 2.1. On Jupiter, a day lasts 9.9 Earth hours and the equatorial circumference is 448,600 km. Knowing that the measured gravitational acceleration at the equator is 26.4 m/s^2, deduce the true gravitational acceleration and the centrifugal acceleration (Cushman-Roisin and Beckers: Physical problems 2-1) .

Answer: Along the Jupiter equator, the outward centrifugal force is vertical and upward. Thus, it directly subtracts from the gravitational force, and the measured gravitational acceleration results from the difference:

$$g_{measured} = g_{true} - g_{centrifugal}$$

Where $g_{measured}$ is 26.4 m/s^2 , g_{true} and $g_{centrifugal}$ are to ber determined. Since Ω = 2л/(9.9hours) = 1.763 10^{-4} s^{-1} and R = (448 600 km)/ 2л = 71 397 km. We find:

$g_{centrifugal} = \Omega^2 R$=2.22 m/s^2

$g_{true} = g_{measured} + g_{centrifugal} = 28.6 \text{ m/s}^2$

Problem A 2.2. The Japanese Shinkansen train (bullet train) zips from Tokey to Osaka (both at approximately 35^{o}N) at a speed of 185 km/h. In the design of the train and tracks, do you think that engineers had to worry about Earth's rotation? (Hint: The Coriolis effect induces a transverse force, which could produce a tendency for the train to lean sideways, Cushman-Roisin and Beckers: Physical problems 2-2).

Answer: At 35^{o}N, the value of the Coriolis parameter is:

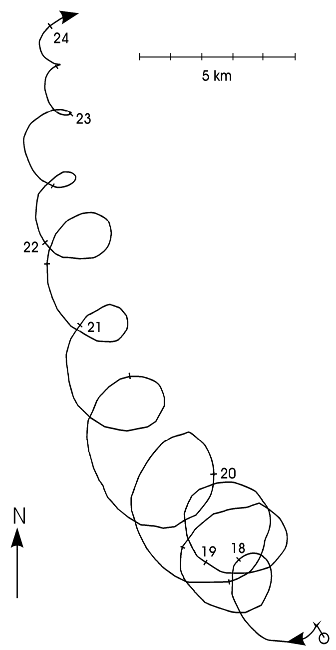
$f = 2\ \Omega \sin 35^o = 2(7.29\ 10^{-5}\ s^{-1}) \sin 35^o = 8.37\ 10^{-5}\ s^{-1}$

The transverse Coriolois acceleration caused by the speed U = 185 km/hr = 51 m/s is

$fU = = (8.37\ 10^{-5}\ s^{-1})\ (51\ m/s) = 4.26\ 10^{-3}\ m/s^2$

Combined with the downward gravitational acceleration of 9.81 m/s^2, this results in a tilt of the net acceleration of 2.5 10^{-2} degrees from the vertical (tilt = arctan (4.26 10^{-3}/9.81) = 2.5 10^{-2}). For the engineer, this is not worrisome.
The vertical Coriolis acceleration, propotional to f_* is in the vertical and causes absolutely no problem. Furthermore, it is not much larger.

Problem A 2.3. Inertia oscillations were observed in the Baltic Sea by Gustafson and Kullenberg in the 1930th . The measurements illustrated in the figure illustrated a damped oscillation. Estimate the damping and outline a model that could simulate the damped oscillation. Also discuss possible mechanism that could cause the damping. See text in Figure 2-10 Cushman-Roisin and Beckers (2005, page 50) (Omstedt, 2005).



Answer: The trajectory illustrate an oscillation with a veering to the right with a period of 14 hours. After about 2-3 days the oscillation is damped. Horizontal, unforced motions are described by Equations 2-22a and 2-22b. Adding a damping term the equation can be written as:

$$\frac{\partial u}{\partial t} = fv - c_{decay}u$$

$$\frac{\partial v}{\partial t} = -fu - c_{decay}v$$

Where c_{decay} is the damping term with the dimension s^{-1}.

If the equations are derivated with respect to time and complex theory is introduced we can solve the equations analytical. With w = u+iv and i^2=-1 the and adding the equations together we find that:

$$\frac{\partial w}{\partial t} + (c_{decay} + if)w = 0$$

Which can be analytical solved as:

$$w = w_0 e^{-c_{decay}t} e^{-ift}$$

The solution thus consists of a damping term with the damping time scale equal to $(c_{decay})^{-1}$ and oscillating time scale equal to $(f)^{-1}$. Based on a slab model also including winds Liljebladh and Stigebrandts estimated c_{decay} to 1/(32 hours). They explained the damping by internal wave drag.

Chapter 3.

Problem A 3.1. Consider the Mediterranean Sea of surface S= $2.5\text{x}10^{12}$ m^2 over which an average heat loss of 7 W/ m^2 is observed. Due to an average surface water loss of 0.9 m/year (evaporation more important than rain and river runoff combined), salinity would increase, water level drop and temperature decrease, if it were not for a compensation by exchange with the Atlantic ocean through the Strait of Gibraltar. Assume that water, salt and heat content of the Mediterranean do not change over time and that the exchange across Gibraltar is accomplished by a two-layer process, establish sea-wide budgets of volume, salt and heat. Given that the Atlantic inflow is characterized by $T_a = 15.5\ (^oC)$, S_a=36.2 and a volume flow of 1.4 Sv, what are the outflow characteristics? Is the outflow at surface or at bottom (Cushman-Roisin and Beckers, 2008: Physical problems 3-4)?

Answer:

Assuming no time dependency we can write the volume, salt and heat conservation according to :

$$Q_o = Q_i + A_s(P-E) + Q_r = Q_i - Q_{ev}$$
$$S_m Q_o = S_a Q_i$$
$$F_o = F_i - A_s F_{loss}$$

From volume conservation we find that:

$$Q_o = Q_i - Q_{ev} = 1.4 \text{ Sv-}\frac{0.9*2.5x10^{12}}{365*24*3600} = 1.4\text{Sv-}0.07\text{Sv}=1.33\text{Sv}$$

From salt conservation we find that:

$$S_m = S_a Q_i / Q_o = 36.2*1.4/1.33 = 38.10$$

From heat conservation we find that

$F_o = F_i - A_s F_{loss}$ or

$$\rho c_p T_m Q_o = \rho c_p T_a Q_i - A_s F_{loss}$$

With the water density ρ equal 1000 kg/ m^3, the specific heat of water c_p equal to 4200 J/(kg ^{o}C) we get:

$$T_m = \frac{\rho c_p T_a Q_i}{\rho c_p Q_o} - \frac{A_s F_{loss}}{\rho c_p Q_o} = \frac{1000x4200x15.5x1.4}{1000x4200x1.33} - \frac{2.5x10^{12}7}{1000x4200x1.33x10^6} = 16.32 - 3.13 = 13.2$$

The Mediterranean bulk temperature become lower than the Atlantic Ocean and the salinity higher, indicating that the outflowing water are at sill bottom.

Chapter 4.

Problem A 4.1. Geophysical flows on the Earth's ranges length and time scales over several order of magnitude. When constructing mathematical models the equation can often only be resolved within a limited resolution. Dynamic features larger than the grid domain then need to be prescribed, while features below the grid size needs to be parameterized. If you were to construct a numerical model of ocean fronts what processes and scales would you paramerize, resolve and prescribe (Omstedt, 2005)?

Answer: From Figure 1-7 below we can learn that the typical length and time scales of ocean fronts are L = 1-10 km and T = 1 to 10 days. Our model thus needs to resolve these scales. A general rule is that one need to have 10 grid cells to resolve a sinus way. This implies that we need to have a grid resolution of 0.1 km and a time resolution of about 2 hours. The model domain is a question of computer capability but let's say that we make the model one order of magnitude larger than the studied process. Thus the model domain is then 100 km and simulation time 100 days. Processes less than 0.1 km and with time scales less tha 2 hours need to be parameterized. That means all aspects of turbulence and rapid changing winds need to be parameterixed. Processes larger than 100 km and longer than 100 days as gestrophic eddies or mean currents needs to be prescribed with e.g. constant values.

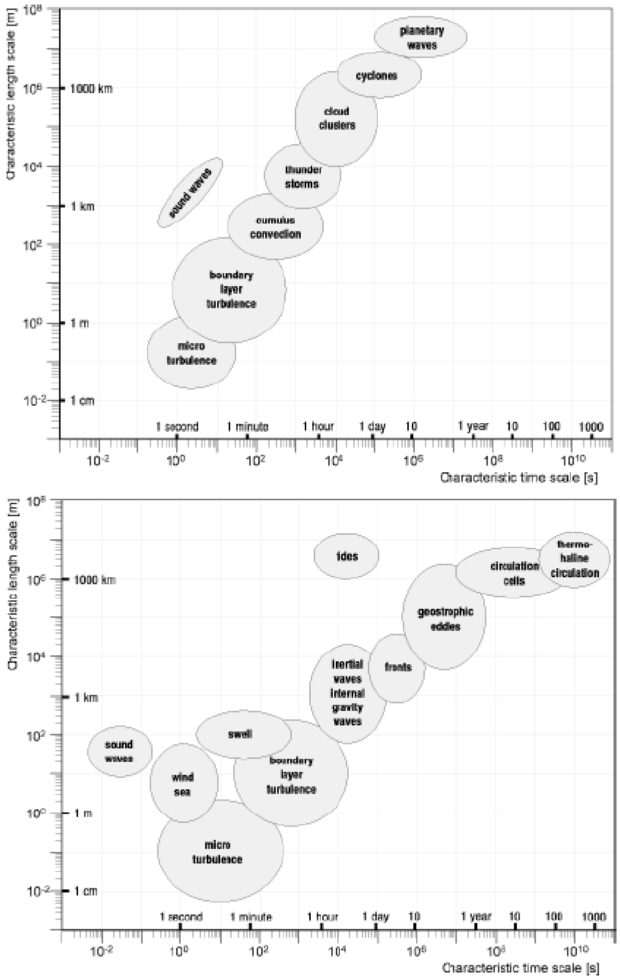


Figure 1-7 Various types of processes and structures in the atmosphere (top panel) and oceans (bottom panel), ranked according to their respective length and time scales. (Diagram courtesy of Hans von Storch)

Problem A 4.2. Using the scaling given in (4-16), compare the dynamic pressure induced by the Gulf Stream (speed = 1 m/s, with 40 km, and depth = 500 m) with the main hydrostatic pressure due to the weight of the same water depth. Also, convert the dynamic pressure scale to its equivalent height of hydrostatic pressure (head). What can you infer about the possibility of measuring oceanic dynamic pressures by pressure gauge (Cushman-Roisin and Beckers: Physical problems 4-2)?

Answer: For the Gulf Stream system the dynamic pressure is

$$P = \rho_0 \Omega L U = (1000 kg/m^3)(7.2910^{-5} s^{-1})(410^4 m)(1 m/s) = 2.910^3 N/m^2$$

Whereas the hydrostatic pressure is

$$P_H = \rho_0 g H = (1000 kg/m^3)(9.81 m/s^2)(500 m) = 4.910^6 N/m^2$$

The latter is obviously much larger than the former (factor of more than 1600!).
To convert the dynamic pressure scale into equivalent height, say h, of hydrostatic pressure, we write:

$$P = \rho_0 \Omega L U = \rho_0 g h$$

Or

$$h = \frac{\Omega L U}{g} = \frac{(7.2910^{-5} s^{-1})(410^4 m)(1m/s)}{(9.81 m/s^2)} = 0.30m$$

This implies that the pressure signal of the mighty Gulf Stream is equivalent to only a 30 cm variation in sea level elevation. Since this is small compared to incessant wave action, we conclude that inferring the dynamic pressure, and hence the geostrophic currents, from direct pressure measurements is doomed to failure. (In practice, oceanographers measure the density field and calculate pressure from that.)

Chapter 7.

Problem A 7.1. A laboratory experiment is conducted in a cylindrical tank 20 cm diameter, filled with homogeneous (15 cm deep at the center) water and rotating at 30 rpm. A steady flow field with maximum velocities of 1 cm/s is generated by a source-sink device. The water viscosity is 10^{-6} m^2/s. Verify that this flow field meets the conditions of geostrophy (Cushman-Roisin and Beckers: Physical problems 7-1).

Answer: We need to show that the Rossby number and the Ekman number are all substantially less than unity, i.e.

$$Ro_T = \frac{1}{\Omega T} \ll 1, Ro = \frac{U}{\Omega L} \ll 1, Ek = \frac{\nu}{\Omega H^2} \ll 1$$

Here, the rotation rate Ω is equal to 30 rpm = 3.14 s^{-1} (one revolution = 2 π, one minute = 60s), and the kinematic viscosity is viscosity is 10^{-6} m^2/s.

Since it may be assumed that the source-sink flow is steady in the rotating framework, T is infinity. The other scales are obviously U = 0.01 m/s, L = 0.1 m and H = 0.15 m. From this, we find that:

$$Ro_T = 0, Ro = 0.032, Ek = 1.410^{-5}$$

Since all these quantities are substantially less than unity, the flow must be very nearly geostrophic.

Problem A 7.2. Demonstrate the assertion made at the end of Section 7-2, namely, that the geostrophic flows between irregular bottom and top boundaries are constrained to be directed along lines of constant fluid depth (Cushman-Roisin and Beckers: Physical problems 7-3).

Answer: If the bottom boundary is z=b(x,y) and the top boundary is z=a(x,y), with a>b everywhere, the boundary conditions are:

$$Bottom(z=b): w = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}$$

$$Top(z=a): w = u\frac{\partial a}{\partial x} + v\frac{\partial a}{\partial y}$$

Since in geostrophic flow the vertical derivates of the velocity components vanish, the vertical velocity must be same at both boundaries:

$$\left[u\frac{\partial a}{\partial x} + v\frac{\partial a}{\partial y}\right]_{z=a} = \left[u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}\right]_{z=b}$$

Since u and v, too, are the same at the boundaries:

$$u\frac{\partial}{\partial x}(a-b) + v\frac{\partial}{\partial x}(a-b) = 0$$
$$or$$
$$\vec{V_h} \cdot \nabla H = 0$$

Where H = a-b is the local fluid depth. Hence, the geostrophic flow is directed along lines of constant fluid depth.

Problem A 7.3. As depicted in Figure 7-13, a vertically uniform but laterally sheared coastal current must climb a bottom escarpment. Assuming that the jet velocity still vanishes offshore, determine the velocity profile and the width of the jet downstream of the escarpment using h_1= 200 m, h_2= 160 m, U_1= 0.5 m/s, L_1= 10 km and f = 10^{-4} s^{-1}. What would happen if the downstream depth were only 100 m (Cushman-Roisin and Beckers: Physical problems 7-6)?

Answer: The upstream current has a cyclonic vorticity equal to U_1/L_1 and the potential vorticity is thus

$$q = \frac{f + \frac{U_1}{L_1}}{h_1} = \frac{(10^{-4} + 510^{-5})s^{-1}}{200m} = 7.5010^{-7} m^{-1} s^{-1}$$

Downstream of the escarpment, the velocity has changed to U_2/L_2 (where neither U_2 or L_2 is known), but the potential vorticity has remained unchanged. Thus,

$$\frac{f+\frac{U_2}{L_2}}{h_2}=\frac{f+\frac{U_1}{L_1}}{h_1}$$

This yields one relationship between the two unknowns. The other relationship is derived from conservation of fluid transport:

$$\frac{1}{2}U_2L_2h_2=\frac{1}{2}U_1L_1h_1$$

Eliminating L_2 between these two equations provides:

$$U_1^2-U_2^2=fU_1L_1\frac{h_1-h_2}{h_2}$$

Since h_1 is larger than h_2 it follows that U_2 must be less than that U_1. The numerical value is

$$U_2=\left[U_1^2-fU_1L_1\frac{h_1-h_2}{h_2}\right]^{\frac{1}{2}}=\left[0.25m^2/s^2-0.125m^2/s^2\right]^{\frac{1}{2}}=0.35m/s$$

From conservation of fluid transport it follows that the downstream width of the current is:

$$L_2=\frac{U_1L_1h_1}{U_2h_2}=17.7km$$

In the second case, where $h_2 = 100$ m, the square root yields an imaginary number and no U_2 value can be obtained. The source of this problem can be understood by considering the statement of potential vorticity conservation above. If the values given ($f = 10^{-4}$ s^{-1}, $U_1= 0.5$ m/s, $L_1 = 10^4$ m, $h_1 =200$ m and $h_2 =100$ m), we find the downstream vorticity to be

$$\frac{U_2}{L_2}=\frac{h_2U_1}{h_1L_1}-\frac{h_1-h_2}{h_1}f=-2.510^{-5}s^{-1}$$

The negative sign implies anticyclonic vorticity. (The squeezing has been so strong that the cyclonic vorticity has been annihilated and replaced by some anticyclonic vorticity). If the flow were to continue along the coast, its speed would not decrease but instead increase away from the coast, which is highly unrealistic. The obvious conclusion is that the flow cannot continue along the wall but must instead turn offshore and flow along the escarpment, stradding its edge. On the left of such offshore current, the water is deeper ($h_1 =200$ m) and the vorticity is cyclonic, while on the right the water is shallow ($h_2 =100$ m) and the vorticity is anticyclonic. Such a vorticity distribution is compatible with a triangular jet-like current. The characteristics of that jet can be determined.

If we denote by U the velocity maximum at the peak, right over the escapment, by L_d the width over the deep ledt side (200 m) and L_s the width over the shallow right side, we have the following equations:

$$deep-water-vorticity: \frac{U}{L_d} = \frac{U_1}{L_1} = +5 10^{-5} s^{-1}$$

$$shallow-water-vorticity: -\frac{U}{L_s} = \frac{h_2 U_1}{h_1 L_1} - \frac{h_1 - h_2}{h_1} f = -2.5 10^{-5} s^{-1}$$

$$volume-flux-conservation: \frac{1}{2} U L_d h_1 + \frac{1}{2} U L_s h_2 = \frac{1}{2} U_1 L_1 h_1 = 5 10^5 m^3 s^{-1}$$

The solution is:

U = 0.35 m/s, L_d = 7.07 km, L_s = 14.1 km.

Problem A 7.4. What are the differences in dynamic pressure across the coastal jet of Problem 7-6 upstream and downstream of the escarpment? Take h_2 =160 m and ρ_o = 1022 kg/m^3 (Cushman-Roisin and Beckers: Physical problems 7-7).

Answer: If the x-axis is aligned with the coast in the downstream direction and the y-axis directed offshore, then the flow field takes the form

$$u = U(1 - \frac{y}{L}), v = 0$$

Where y=0 is the coast. Upstream of the escarpment the parameter values are U = U_1 = 0.5 m/s, L= L_1 = 10^4 m and, downstream, U = U_2 = 0.35 m/s, L= L_2 = 1.77$10^4$ m. (See answer to Problem 4-6).

Geostrophy requires

$$+fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

And yields the pressure distribution across the jet:

$$p = P - \rho_0 f U (y - \frac{y^2}{2L})$$

Where P is an unimportant constant of integration. The off-shore pressure difference (the net dynamic pressure) is

$$\Delta p = p(y=0) - p(y=L) = \frac{1}{2}\rho_0 f U L$$

Upstream, the value is

$$\Delta p_u = \frac{1}{2}(1022kg/m^3)(10^{-4}s^{-1})(0.5m/s)(10^4 m) = 255.5N/m^2$$

While downstream it is

$$\Delta p_d = \frac{1}{2}(1022kg/m^3)(10^{-4}s^{-1})(0.35m/s)(1.7710^4 m) = 319.4N/m^2$$

Since the offshore pressure is uniform, the pressure along the coast increased by 319.4-255.5=63.9 N/m^2 downstream of the escarpment. Because the flows up-pressure, it must be decelerated. This can be verified by comparing the downstream and upstream momentum fluxes:

$$momentum - flux = \rho_0 \int_0^L u^2 H dy = \frac{1}{3}\rho_0 HLU^2$$

$$upstream - momentum - flux = \frac{1}{3}\rho_0 h_1 L_1 U_1^{\ 2} = 1.7010^8 N$$

$$downstream - momentum - flux = \frac{1}{3}\rho_0 h_2 L_2 U_2^{\ 2} = 1.2010^8 N$$

The momentum flux is indeed less downstream than upstream.

Chapter 8.

Problem A 8.1. Assume that the atmospheric Ekman layer over the earth's surface at latitude 45 ^{o}N can be modelled with a turbulent kinematic viscosity υ = 10 m^2/s. If the geostrophic velocity above the layer is 10 m/s and is uniform, what is the vertically integrated flow across the isobars (pressure contours)? Is there any vertical velocity (Cushman-Roisin and Beckers: Physical problems 8-2)?

Answer: At 45 ^{o}N, the Coriolis parameter is

$$f = 2\Omega \sin\phi = 2(7.2910^{-5}s^{-1})\sin(45^o) = 1.0310^{-4}s^{-1}$$

With the given value of υ, the Ekman depth is

$$d = \left(\frac{2\nu}{f}\right)^{1/2} = \left(\frac{2(10m^2/s)}{1.0310^{-4}s^{-1}}\right)^{1/2} = 440m$$

Under a geostrophic mean flow of 10 m/s, the cross-isobaric Ekman transport is; according to (8-19):

$$V = \left(\frac{\bar{u}d}{2}\right) = \frac{(10m/s)(440m)}{2} = 2200m^2/s$$

There is no vertical velocity because a uniform flow has no curl.

Problem A 8.2. You are working for a company that plans to deposit high-level radioactive wastes on the bottom of the ocean, at a depth of 3000 m. This site (latitude 35 ^{o}N) is known to be at the center of a permanent counterclockwise vortex. Locally, the vortex flow can be assimilated to a solid-body rotation with the angular speed equal to 10^{-5} s^{-1}. Assuming a homogeneous ocean and a steady, geostrophic flow, estimate the upwelling rate at the vortex center. How many years will it take for the radioactive vastes to arrive at the surface? Take f = 810^{-5} s^{-1} and $\upsilon = 10^{-2}$ m^2/s (Cushman-Roisin and Beckers: Analytical Problems 8-5).

Answer: Noting $\Omega = 10^{-5}$ s^{-1} as the angular rotation rate of the vortex flow, we write the velocity components of the flow away from the bottom:

From vector products:

$$\vec{a} \times \vec{b} = \vec{i}(a_y b_z - a_z b_y) - \vec{j}(a_x b_z - a_z b_x) + \vec{k}(a_x b_y - a_y b_x)$$

or

$$\vec{V} = \vec{\Omega} \times \vec{r} = (0,0,\Omega) \times (x,y,z) = -\vec{i}\Omega y + \vec{j}\Omega x$$

We then can write the the velocity components outside the bottom boundary layer:

$$\bar{u} = -\Omega y$$
$$\bar{v} = +\Omega x$$

Equation (8-20) then provides the vertical velocity in the interior

$$\bar{w} = \frac{d}{2}\left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right) = \frac{d}{2} 2\Omega = \sqrt{\frac{2\upsilon}{f}}\Omega = (15.8m)(10^{-5}s^{-1}) = 1.5810^{-4} m/s = 13.7m/day$$

Waste from the bottom, at a depth H=3000 m, will be brought to surface in

$$upwelling - time = \frac{3000m}{13.7m/day} = 220days$$

That is, in about 7 to 8 months.

Problem A 8.3. Between 15 ^{o}N and 45 ^{o}N, the winds over the North Pacific consists mostly of the easterly trades (15 ^{o}N to 30 ^{o}N) and the westerlies (30 ^{o}N to 45 ^{o}N). An adequate representation is

$$\tau^x = \tau_0 \sin(\frac{\pi y}{2L}), \tau^y = 0, -L \le y \le L$$

Where τ_0 = 0.15 N/m^2 is the maximum wind stress and L = 1670 km. Taking ρ_0 = 1028 kg/m^3 and the value of the Coriolis parameter corresponding to 30 ^{o}N, calculate the Ekman pumping. Which way is it directed? Calculate the vertical volume flux over the entire 15 ^{o} - 45 ^{o}N strip of the North Pacific (width = 8700 km). Express your answer in Sverdrup units (1 Sverdrup = 1 Sv = 10^6 m^3/s, Cushman-Roisin and Beckers: Physical problems 8-6).

Answer: Using equation (8-36) we obtain directly the Ekman pumping over the mid-latitude North Pacific (15 ^{o}N and 45 ^{o}N)

$$\overline{w} = \frac{1}{\rho_0 f}\left(\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right) = -\frac{\pi \tau_0}{2\rho_0 f L}\cos(\frac{\pi y}{2L})$$
$$-L \le y \le L$$

With the given values and f according to

$$f = 2\Omega \sin\phi = 2(7.29 10^{-5} s^{-1})\sin(30^o) = 7.29 10^{-5} s^{-1}$$

We obtain

$$\overline{w} = -1.88 10^{-6}\cos(9.41 10^{-7} y)$$

With y in m and w in m/s. Over the defined y-range, the cosine function is positive and the vertical velocity negative. There is thus a downward Ekman pumping over these latitudes. The maximum absolute value (at y = 0, 30 ^{o}N) is 1.88 10^{-6} m/s or 59 m/year.

The volume flux caused by this Ekman pumping is obtained by integration over the domain

$$Volume-flux = \iint \left|\overline{w}(y)\right| dxdy = a\frac{\pi \tau_0}{2\rho_0 f L}\int_{-L}^{L}\cos(\frac{\pi y}{2L})dy$$

Where a = 8700 km = 8.70 10^6 m is the zonal extent of the basin. The integration yields:

$$Volume-flux = \frac{2a\tau_0}{\rho_0 f} = \frac{2(8.70 10^6 m)(0.15 N/m^2)}{(1028 kg/m^3)(7.29 10^{-5} s^{-1})} = 3.48 10^7 m^3/s = 34.8 Sv$$

Chapter 9.

Problem A 9.1. Because the Coriolos parameter vanishes along the equator, it is usual in the study of tropical processes to write

$$f = \beta_0 y$$

Where y is the distance measured from the equator (positive northward). The linear wave equations the take the form

$$\frac{\partial u}{\partial t} - \beta_0 yv = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + \beta_0 yu = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Where u and v are the zonal and meridional velocity components, η is the surface displacement, g is gravity, and H is the ocean depth. Explore the possibility of a wave travelling zonally with no meriodal velocity. At which speed does this wave travel and in which direction? Is it trapped along the equator? If so, what is the trapping distance? Does this wave bear any resemble to a midlatitude wave (f not zero, Cushman-Roisin and Beckers: Physical problems 9-6)?

Answer: If there is no meriodal velocity (v=0), the equations reduce to

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$
$$\beta_0 yu = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

The first and last equations are identical to those in the Kelvin-wave theory, except for a permutation of coordinates, and admit the solutions

$$u = U_1(x+ct, y) + U_2(x-ct, y)$$
$$\eta = -\sqrt{\frac{H}{g}}U_1(x+ct, y) + \sqrt{\frac{H}{g}}U_2(x-ct, y)$$

Where $c = \sqrt{gH}$ is the zonal wave speed. Substitution of this solution in the remaing equation yields

$$\frac{\partial U_1}{\partial y} = \frac{\beta_0}{c} yU_1$$
$$\frac{\partial U_2}{\partial y} = -\frac{\beta_0}{c} yU_2$$

Solving for the meridional structure of these waves, we find that the U_1 function grows exponentially away from the equator and must be rejected. By contrast, the U_2 function decays away from the equator according to

$$U_2 = U\,\mathrm{e}^{-\frac{\beta_0 y^2}{2c}}$$

Where U is an arbitary function of (x-ct). The final solution is:

$$u = U(x-ct)\,\mathrm{e}^{-\frac{\beta_0 y^2}{2c}}$$
$$v = 0$$
$$\eta = \sqrt{\frac{H}{g}}U(x-ct)\,\mathrm{e}^{-\frac{\beta_0 y^2}{2c}}$$

This solution corresponds to an eastward propagating wave with maximum amplitude along the equator and decaying symmetrically away from it in both hemispheres. Because of its decay away from the equator, the wave can be said to be trapped; the trapping distance is provided by the values of y for which the exponent reaches unity (e^{-1} folding scale), which is

$$trapping - scale = \sqrt{\frac{2c}{\beta_0}}$$

The wave is non-dispersive and propagates at the speed $c = \sqrt{gH}$ equal to that of a surface gravity wave.

In view of the above properties, this wave is obviously the equatorial analogue of the coastal Kelvin wave. In a sense, the equator acts as a wall (v=0), and we have two Kelvin waves, one in each hemisphere and both propagating eastward, the northern one having the “wall” on its right and the southern one having the “wall” to its left.

Chapter 10.

Problem A 10.1. What can you say of the stability properties of the following flow fields on the f-plane (Cushman-Roisin and Beckers: Physical problems 10-2)?

$$\bar{u}(y) = U\left(1 - \frac{y^2}{L^2}\right), \left(-L \le y \le L\right)$$
$$\bar{u}(y) = U \sin(\frac{\pi y}{L}), \left(0 \le y \le L\right)$$
$$\bar{u}(y) = U \cos(\frac{\pi y}{L}), \left(0 \le y \le L\right)$$
$$\bar{u}(y) = U \tanh(\frac{y}{L}), \left(-\infty \prec y \prec \infty\right)$$

Answer: According to the theory presented in Section 10-2, necessary conditions for instability on the f-plane ($\beta_0 = 0$) are:

1. $\frac{d^2\bar{u}}{dy^2}$ vanishes at least once in the domain,

2. $(\bar{u}-\bar{u}_0)(\frac{d^2\bar{u}}{dy^2})$ is negative in some portion of the domain, where $\bar{u}_0$ is the value of $\bar{u}$ where $\frac{d^2\bar{u}}{dy^2}$ vanishes. We therefore have to calculate the second derivate of each velovity profile.
We obtain

$$\frac{d^2\bar{u}}{dy^2} = -\frac{2U}{L^2}, (-L \le y \le L)$$
$$\frac{d^2\bar{u}}{dy^2} = -\frac{\pi^2 U}{L^2}\sin(\frac{\pi y}{L}), (0 \le y \le L)$$
$$\frac{d^2\bar{u}}{dy^2} = -\frac{\pi^2 U}{L^2}\cos(\frac{\pi y}{L}), (0 \le y \le L)$$
$$\frac{d^2\bar{u}}{dy^2} = -\frac{2U}{L^2}\frac{\sinh(y/L)}{\cosh^3(y/L)}, (-\infty \prec y \prec \infty)$$

The second derivate of the first profile vanishes no where, while the second profile vanishes at the boundaries. Therefore, these profiles do not meet condition (1) and are stable (in the absence of viscosity). The second derivates of the third and fourth profiles vanish at y=L/2 and y=0, respectively. There, the velocity $\bar{u} = \bar{u}_0$ is zero. Condition (1) is met, and condition (2) asks us to check whether the expressions

$$-\frac{\pi^2 U^2}{L^2}\cos^2(\frac{\pi y}{L})$$
$$-\frac{2U^2}{L^2}\frac{\sinh^2(y/L)}{\cosh^4(y/L)}$$

may be negative in their respective domains. They indeed are negative throughout their domains, and condition (2) is satified for each profile.We concludes that the third and fourth profiles meet the necessary conditions for instability and are likely to be unstable, although we cannot be sure.

Chapter 20 (Chapter 8 on 1994 book).

Problem A 20.1. From the Stommel theory of the western boundary current in the large-scale oceanic circulation, derive analytical expressions for the scale of the width of the boundary layer and the scale of the current speed within it. Derive these scales using the following quantities: Coriolis-parameter mean value and gradient f_0 and β_0, mean ocean depth H, zonal width of basin L_1, zonal wind-stress amplitude τ_0, meridional distance between two latitudes where the wind-stress curl vanishes L_2 , reference water density ρ_0 , and Ekman-layer depth d (Cushman-Roisin, 1994, Problem 8-1).

Answer: From the expression (8-20 Now 20.36c) giving the meridional velocity

$$v = \frac{1}{\rho_0 \beta_0 H} \frac{d\tau^x}{dy} \left[`-1 + \frac{2\beta_0 H L_1}{f_0 d} e^{(-\frac{2\beta_0 H}{f_0 d} x)} \right]$$

We note that the exponential decay of the current from its maximum value at the coast to its Sverdrup value in the interior occurs over the e-folding scale:

$$\frac{f_0 d}{2\beta_0 H}$$

This length can be considered as the width of the boundary current. Note that it does not depend on L_1, the width of the basin. The velocity scale is obtained by examining the value of v at the coast

$$v_{coast} = \frac{1}{\rho_0 \beta_0 H} \frac{d\tau^x}{dy} \left[\frac{2\beta_0 H L_1}{f_0 d} \right] = \frac{2L_1}{\rho_0 f_0 d} \frac{d\tau^x}{dy}$$

Estimating the wind-stress $\frac{d\tau^x}{dy}$ at $\frac{\tau_0}{2L_2}$ (the factor 2 is optional), we derive the following velocity scale

$$V = \frac{L_1 \tau_0}{\rho_0 f_0 d L_2}$$

Note that it is inversely proportional to the small Ekman depth d (and is therefore large) and is proposional to the basin width L_1. This last conclusion could have been anticipated: The wider the basin, the greater the southward Sverup flow that must be returned northward by the boundary current.

Problem A20.2 Given that the North Pacific Ocean is about twice as wide as the North Atlantic Ocean and that both basins are subjected to equally strong winds, compare their boundary-layer widths and boundary-current speeds Cushman-Roisin and Beckers: Physical problems 8-3).

Answer: For the Pacific Ocean, the basin width L_1 is about twice that of the Atlantic Ocean, while all other parameters are about the same. According to the scales derived in Problem A8.1 above, we conclude that the two oceans share the same boundary layer width (independent of L_1), and that the velocity scale for the Pacific is about twice as large as that for the Atlantic (propotional to L_1). This is because, in a basin twice as large, there is twice as much Sverdrup flow to be returned by the boundary current.

Chapter 11.

Problem A 11.1. The Gulf Stream waters are characterized by surface temperatures around 22^o C. At a depth of 800 m below the Gulf Stream, temperature is only 10^o C. Using the value

2.1 10^{-4} K^{-1} for the coefficient of thermal expansion, calculate the stratification frequency. What is the horizontal length scale at which rotation and stratification play comparable roles? Compare this length scale to the width of the Gulf Stream (Cushman-Roisin and Beckers: Physical problems 11-1).

Answer: Approximating the definition (11-3) of the stratification frequency by taking density differences over the top 800 m and using the equation of state $\rho = \rho_0(1-\alpha(T-T_0))$, we obtain:

$$N^2 = -\frac{g}{\rho_0}\frac{\Delta\rho}{H} = +\frac{\alpha g \Delta T}{H} = \frac{(2.1 10^{-4}K^{-1})(9.81ms^{-2})((22+273)K-(10+273)K)}{(800m)} = 3.09 10^{-5}s^{-2}$$

Or N = 0.00556 s^{-1} . The corresponding period of oscillation is 2π/N = 1130 s = 19 min. The horizontal length at which rotation and stratification play comparable role is given by (11-28):

$$L = \frac{NH}{\Omega} = \frac{(0.00556s^{-1})(800m)}{(7.29 10^{-5}s^{-1})} = 60985m$$

This is comparable to the width of the Gulf Stream, observed to be around 80 km.

Chapter 13.

Problem A 13.1. Derive the dispersion relation of internal gravity waves in the presence of rotation, assuming f is less then N. Show that the frequency of these waves always must be higher than f but lower than N. Compare vertical phase speed to vertical group velocity (Omstedt, 2005).

Answer: We first write down the governing equations for internal gravity waves with rotation.

$$(I): \frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$(II): \frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$(III): \frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{1}{\rho_0}g\rho'$$

$$(IV): \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$(V): \frac{\partial \rho'}{\partial t} + w\frac{\partial \overline{\rho}}{\partial z} = 0$$

We have 5 equations and 5 unknowns $u, v, w, p' and \rho'$. We will first derivate one equation describing the vertical velocity and then assume a wave solution from which we can calculate the dispersion relation.

$$\frac{\partial}{\partial x}(I) + \frac{\partial}{\partial y}(II) \Rightarrow$$

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - f(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) = -\frac{1}{\rho_0}(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}) = -\frac{1}{\rho_0}\nabla_h^2 p'$$

$$with(IV) \Rightarrow$$

$$(VI): \frac{\partial}{\partial t}(\frac{\partial w}{\partial z}) + f(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) = \frac{1}{\rho_0}\nabla_h^2 p'$$

$$\frac{\partial}{\partial y}(I) - \frac{\partial}{\partial x}(II) \Rightarrow$$

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}) - f(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$$

$$with(IV) \Rightarrow$$

$$(VII): \frac{\partial}{\partial t}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) - f(\frac{\partial w}{\partial z}) = 0$$

$$\frac{1}{f}\frac{\partial}{\partial t}(VI) - (VII) \Rightarrow$$

$$(VIII): \frac{1}{f}\frac{\partial^2}{\partial t^2}(\frac{\partial w}{\partial z}) + f(\frac{\partial w}{\partial z}) = \frac{1}{f\rho_0}\frac{\partial}{\partial t}\nabla_h^2 p'$$

$$\frac{\rho_0}{g}\frac{\partial}{\partial t}(III) - (V) \Rightarrow$$

$$(IX): (\frac{\partial^2}{\partial t^2} + N^2)w = -\frac{1}{\rho_0}\frac{\partial}{\partial t}(\frac{\partial p'}{\partial z})$$

$$f\frac{\partial}{\partial z}(VIII) + \nabla_h^2(IX) \Rightarrow$$

$$(X): \frac{\partial^2}{\partial t^2}\nabla^2 w + N^2\nabla_h^2 w + f^2\frac{\partial^2 w}{\partial z^2} = 0$$

The Equation (X) is now our equations in w. We now assume a wave solution or $w = w_0 e^{i(lx+my+nz-\omega t)}$. Some algebra yields the dispersion relation:

$$(XI): \omega^2 = \frac{N^2(l^2+m^2) + f^2 n^2}{l^2+m^2+n^2}$$

Which is some weighted averaged of N^2 and f^2. Rewrite the equation:

$$(l^2+m^2)(\omega^2-N^2)=n^2(f^2-\omega^2)$$

As l^2, m^2, n^2 are all positive we can have two possibilities

$$(a): \omega \succ N \Rightarrow f \succ \omega$$
$$(b): \omega \prec N \Rightarrow f \prec \omega$$

In geophysical flows $\omega \prec N$ and the only possibility is case (b) or

$$f \prec \omega \prec N$$

The vertical phase speed is

$$c_z = \frac{\omega}{n}$$

While the vertical group velocity is
$$c_{gz} = \frac{\partial \omega}{\partial n}$$

We need to derivate equation XI. We first derivate the equation side by side:

$$Left = \frac{\partial \omega^2}{\partial n} = \frac{2\omega \partial \omega}{\partial n}$$
$$Right = \frac{\partial}{\partial n}\left(\frac{N^2(l^2+m^2)+f^2n^2}{l^2+m^2+n^2}\right) = \frac{\left(l^2+m^2+n^2\right)2nf^2 - 2n\left(N^2(l^2+m^2)+f^2n^2\right)}{\left(l^2+m^2+n^2\right)^2}$$
$$Right = \frac{2n\left[\left(l^2+m^2+n^2\right)f^2 - \left(N^2(l^2+m^2)+f^2n^2\right)\right]}{\left(l^2+m^2+n^2\right)^2} = \frac{2n\left[\left(l^2+m^2\right)f^2 - N^2(l^2+m^2)\right]}{\left(l^2+m^2+n^2\right)^2}$$

Putting the two sides together reads:

$$c_{gz} = \frac{\partial \omega}{\partial n} = \frac{n(l^2+m^2)(f^2-N^2)}{\omega(l^2+m^2+n^2)^2}$$

Their product

$$c_z c_{gz} = \frac{l^2+m^2}{(l^2+m^2+n^2)^2}(f^2-N^2)$$

Has the sign of $(f^2 - N^2)$. Since f is typically less than N, the sign is negative and the signs of c_z and c_{gz} are opposite. From this we conclude that phase and energy propagate in opposite vertical directions (as in the absence of rotation).

Problem A 13.2. Internal waves are generated along the coast of Norway by the M_2 surface tide (period of 12.42 h). If the buoyancy frequency N is 2 10^{-3} s^{-1}, at which possible angles can the energy propagate with respect to the horizontal? (Hint: Energy propagates in the direction of the group velocity, Cushman-Roisin and Beckers: Physical problems 13-2).

Answer: Since the internal waves are forced by the tides (and hence are called internal tides), their frequency is that of the tide, i.e.

$$\omega = \frac{2\pi}{12.42h} = 1.40510^{-4}(s^{-1})$$

According to equation (13-6), the angel θ between the wavenumber vector and the horizon is

$$\theta = \cos^{-1}(\frac{\omega}{N}) = \cos^{-1}(\frac{1.40510^{-4}s^{-1}}{210^{-3}s^{-1}}) = 85.97^o \approx 86^o$$

From (13-9, 13-10), the direction of energy propagation is given by

$$\tan\alpha = \left|\frac{c_{gz}}{c_{gx}}\right| = \frac{l}{n} = \frac{k\cos\theta}{k\sin\theta} = \frac{1}{\tan\theta} = \tan(90-\theta)$$

In other words, energy propagates at $(90^o - 86^o) = 4^o$ from the horizontal. Both upward and downward propagation is allowed.

Chapter 14.

Problem A 14.1. A stratified shear flow consists of two layers of depth H_1 and H_2 with densities and velocities ρ_1,U_1 and ρ_2,U_2 (left panel of Fig. 14-1). If the lower layer is three times as thick as the upper layer and the lower layer is stagnant, what is the minimum value of the upper-layer velocity for which there is sufficient available kinetic energy for complete mixing (right panel of Fig. 14-2, Cushman-Roisin and Beckers: Physical problems 14-1).

Answer: To determine whether mixing will take place spontaneously or not, we must first evaluate the characteristics of the mixed state:

$$mixed-velocity = U = \frac{H_1U_1 + H_2U_2}{H} = \frac{U_1}{4}$$

$$mixed-density = \rho = \frac{\rho_1 H_1 + \rho_2 H_2}{H} = \frac{\rho_1 + 3\rho_2}{4}$$

Since $H_1 = \frac{H}{4}, H_2 = \frac{3H}{4} and U_2 = 0$

We then calculate the kinetic-energy loss and potential-energy gain caused by the hypothetical mixing:

$$KE_{loss} = \int_0^{H_2} \frac{1}{2}\rho_0 U_2^2 dz + \int_{H_2}^{H} \frac{1}{2}\rho_0 U_1^2 dz - \int_0^{H} \frac{1}{2}\rho_0 U^2 dz$$

$$= \frac{1}{2}\rho_0 U_1^2 \frac{H}{4} - \frac{1}{2}\rho_0 \left(\frac{U_1}{4}\right)^2 H = \frac{3\rho_0 U_1^2 H}{32}$$

$$PE_{gain} = \int_0^{H} \rho g z dz - \int_0^{H_2} \rho_2 g z dz - \int_{H_2}^{H} \rho_1 g z dz$$

$$= \frac{\rho_1 + 3\rho_2}{4} g \frac{H^2}{2} - \rho_2 g \frac{H_2^2}{2} - \rho_1 g \left(\frac{H^2}{2} - \frac{H_2^2}{2}\right) = \frac{3(\rho_2 - \rho_1) g H^2}{32}$$

Mixing proceeds spontaneously if the kinetic-energy loss exceeds the potential-energy gain, i.e. if

$\rho_0 U_1^2 \succ (\rho_2 - \rho_1) g H$

The minimum velocity value for which this conditions is met is

$$U_{1\min} = \sqrt{\frac{(\rho_2 - \rho_1)}{\rho_0} g H} = \sqrt{g' H}$$

Problem A 14.2. In the ocean, a warm current (T=18°C) flows with a velocity of 10 cm/s above a stagnant colder layer (T=10°C). Both layers have identical salinities, and the thermal-expansion coefficient is taken as 2.54 10^{-4} K^{-1}. What is the wave length of the longest unstable wave (Cushman-Roisin and Beckers: Analytical Problems 14-2)?

Answer: We have here a case of Kelvin-Helholtz instability. Criterion (14-5) applied to the case of the equation of state $\rho = \rho_0(1 - \alpha(T - T_0))$, $k = \frac{2\pi}{l}$ provides

$$2\{\rho_0(1 - \alpha(T_2 - T_0)) - \rho_0(1 - \alpha(T_1 - T_0))\} g \prec \rho_0 k \left(U_1 - U_2\right)^2$$

Or

$$2\alpha(T_1 - T_2) g \prec k \left(U_1 - U_2\right)^2$$

The minimum wavenumber for instability is

$$k_{\min} = \frac{2\alpha g(T_1 - T_2)}{(U_1 - U_2)^2} = \frac{(2)(2.5410^{-4}K^{-1})(9.81m/s^2)(8K)}{(0.1m/s)^2} = 3.987m^{-1}$$

Corresponding to the longest unstable wavelength $2\pi / k_{\min} = 15.75m$.

Chapter 19.

Problem A 19.1. Consider the regular gardening greenhouse and idealize the system as follows: The ground and glass act as black bodies (absorbing all the radiation directed towards them), the air play no role, the ground absobs all radiation, and the glass is perfectly transparent to short-wave (visible) radiation and totally opaque to long-wave (heat) radiation. Further, the glass emits its radiation upward and downward in equal parts. Compare the ground temperature inside the greenhouse with that outside. Then, redo the exercise for a greenhouse with two layers of glass separated by a layer of air (Cushman-Roisin and Beckers: Analytical Problems 19-1).

Answer: If E is the amount of long-wave emissions from the ground, the glass (transparent only to short wave radiation) absorbs this long-wave radiation E, re-emitting in two equal portions E/2 upward to space and downward to ground. The ground, therefore, receives the short-wave incident radiation I and the amount E/2 of long-vawe radiation from the glass. Since it emits the amount E, its budget is:

$$E = I + E/2$$

Which implies that E = 2I.
Since without a greenhouse, the budget would have been E = I, we conclude that the presence of a glass panel doubles the amount of radiation emitted by the ground, therefore raising the ground temperature.
If there are two panels of glass separated by a layer of air, we denote by I the amount of short wave incident radiation (absorbed by the ground), E the amount of long-wave radiation emitted by the ground, E_1 by the lower glass panel and , E_2 by the upper glass panel. The budgets (sum of amounts received equals emitted) are:

Ground: $I + E_1/2 = E$
Lower glass: $E + E_2/2 = E_1$
Upper glass: $E_1/2 = E_2$

The solution is $E = 3I$, $E_1 = 4I$, $E_2 = 2I$.

Therefore, with two layers of glass the ground radiation is triple of that it would have been with glass at all. (For n layer of glass, the ground emission is equal to (n+1)I).

Problem A 19.2. Consider the crudest heat budget for the earth (without atmosphere and hydrological cycle) and assume the following dependency of the albedo on temperature: At low temperatures, much ice and clouds cover the earth, yielding a high albedo, whereas at high temperatures, the absence of ice and clouds reduce the albedo to zero. Taking the functional dependence as:

$$\alpha = 0.5 \quad for \quad T \leq 250K$$

$$\alpha = \frac{270 - T}{40} \quad for \quad 250K \leq T \leq 270K$$

$$\alpha = 0 \quad for \quad T \succ 270K$$

Solve for the earth's average temperature T. Discuss the several solutions (Cushman-Roisin and Beckers: Analytical Problems 19-2).

Answer: If I is the amount of short-wave radiation received at the surface, the amount αI is reflected (by definition of the albedo coefficient α) and the remainder, (1-α)I, is absorbed. Equilibrium requires that the amount emitted, σT^4, be equal to that absorbed, and we therefore have:

$$(1-\alpha)I = \sigma T^4$$

Where the albedo as a function of temperature is given in the problem, I = 344 W/m^2.

- If $T \leq 250K$ we have $0.5I = \sigma T^4, \quad or \quad T = 234.7K, \quad or \quad T = -38^o C$.
- If $T \succ 270K$ we have $I = \sigma T^4, \quad or \quad T = 279.1K, \quad or \quad T = +6.1^o C$.
- If $250K \leq T \leq 270K$ we find (after numerical iterations) a third acceptable solution $T = 260.2K, \quad or \quad T = -12.8^o C$.

Physically, we have three realizable states, a coldest state with much ice and much reflection out to space, a warmest state with no ice and no reflection, and an intermediate state. This intermediate state is unstable; indeed, if a small perturbation is introduced to raise the temperature, some ice will melt, the albedo will decrease, the radiation absorbed will increased and the temperature will increase further; similarly, if the temperature is lowered a bit, it will continue to do so. Therefore, the system admits two distinct, stable states, a cold state andd a warm state. (Initial conditions determine towards which state the system evolves).

Using the global heat budget of the earth model, complete with atmospheric layer and a hydrological cycle, explore a worts-case scenario whereby elevated concentrations of greenhouse gases completely block the transmission of long-wave radiation from the earths's surface, the intensity of the hydrological cycle is unchanged, and the anticipated global warming has caused the complete melting of all ice sheets, effectively eliminating all reflection by the earths surface of short-wave solar radiation. What would then be the globally averaged temperature of the earth's surface? (Expect for those transmission and reflection coefficients that need to be revised, use the parameter values quoted in the text, Cushman-Roisin and Beckers: Analytical Problems 19-3).

Answer: We start from the third model in Chapter 19.2. If the emission of long-wave radiation by the earths surface is completely absorbed by the atmospheric greenhouse gases (β_2=0) and if the reflection by earths surface of short-wave radiation is nil (α_2=0), then the radiation budget reduce to:

$$Surface: \quad E_2 = \beta_1 I + 0.64 E_1 - H$$

$$Atmosphere: \quad E_1 = (1 - \alpha_1 - \beta_1) I + E_2 + H$$

$$With: \quad I = 344 W/m^2, H = 113.6 W/m^2, \alpha_1 = 0.33, \beta_1 = 0.49$$

$$We \quad obtain: \quad E_1 = 640.2 W/m^2 \quad and \quad E_2 = 464.7 W/m^2.$$

$$From \quad E_2 = \sigma T^4$$

From $E_2 = \sigma T^4$, we deduce the resulting surface temperature: $T = 300.9K = 27.9^o C$.

3. Problems without solution

Chapter 1.

Problem B 1. 1. Name some naturally occurring flows in the ocean. (Omstedt, 2005).

Problem B 1.2. How did geophysical flows contribute to Christopher Columbus' discovery of the New World and the subsequent exploration of the eastern shore of North America? (Think of both large-scale winds and malor ocean currents, Cushman-Roisin and Beckers: Physical problems 1-2).

Problem B 1.3. When was the optimal month to leave Göteborg and Shanghai for an East Indiamen? (Think of both large-scale winds and major ocean currents, Rodhe, 2005).

Problem B 1.4. Some say that an oceanographers dream is to study the Earth's rotation by sitting in a bath-tub and letting the water go out at the same time as he/she is passing the equator. Is Earth's rotation important for the water flow when emptying a bath tube? Assume the horizontal scale of 1 m, drainage speed on the order of 0.01 m/s, the motion time scale 1000 s and the ambient rotation rate $7.3 \cdot 10^{-5}$ (Omstedt, 2005).

Problem B 1.5.

Chapter 2.

Problem B 2.1. The curve reproduced in Figure 2-19 is a progressive vector diagram constructed from current meter observations at latidude $43^{o}\ 09^{'}$ N in the Mediterranean Sea. Under the assumption of a uniform but time dependent flow field in the vicinity of the mooring, the curve can be interpreted as the trajectory of a water parcel. Using the marks counting the days along the curve, show that this set of observations reveal the presence of inertial oscillations. What is the average orbital velocity in these oscillations (Cushman-Roisin and Beckers: Physical problems 2-10)?

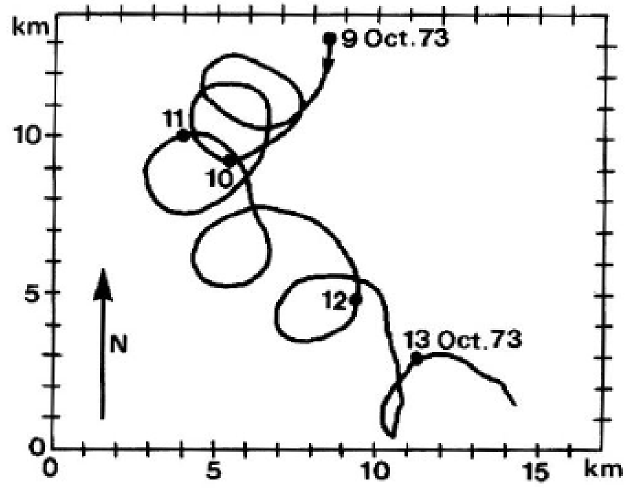


Figure 2-19 Progressive vector diagram constructed from current-meter observation in the Mediterranean Sea taken in October 1973 (Problem 2-10). [Courtesy of Martin Mork, University of Bergen, Norway]

Problem B 2.2. Compare the initia oscillation in the Mediterranean Sea and the Baltic Sea (Figure 2-19 below with Figure in problem A 2.3) with respect to damping. What are the differences and what may this imply? (Omstedt, 2005)

Chapter 3.

Problem B 3.1
The mean depth of the Baltic Sea is 54 m and the surface area 392 978 km^2. How much will the Baltic Sea sea level increase during a year with river waters at 15 000 m^3/s and no outflows? If the out flowing volume flow is 30 000 m^3/s, how large is the inflowing volume flow? If the salinity in the inflowing water is 17 which salinity will the basin have (Omstedt, 2009)?

Problem B 3.2
Consider the Baltic Sea with a surface area of 392 978 km^2. Assume that the volume and heat content of the Baltic Sea do not change over time and that the exchange through the entrance area is accomplished by a two layers flow. Given that the in flowing and the fresh water volumes are equal to 15 000 m^3/s, that the inflowing and out flowing water temperatures are equal to 8 (^{o}C) and that the river runoff do not influence the heat budget. What are the estimated heat loss from the Baltic Sea (Omstedt, 2009)?

Chapter 4.

Problem B 4.1.The equations govering geophysical flows (Eqs. 4.21a – 4.21e) rely on several assumptions. List the assumptions and name some dynamic processes that we can not solve with these equations (Omstedt, 2005).

Problem B 4.2. Geophysical flows on the Earth's ranges length and time scales over several order of magnitude. When constructing mathematical models the equation can often only be resolved within a limited resolution. Dynamic features larger than the grid domain then need to be prescribed, while features below the grid size needs to be parameterized. If you were to construct a numerical model of coastal up-welling what processes and scales would you paramerize, resolve and prescribe (Omstedt, 2005)(Hint: Assume that the typical length and time scales of coastal up-welling are horizontal scales = 10 km, vertical scale = 100 m, and time scale = 10 days, that your model domain is 10 times as large both in space and time and that you need to have gridsizes that are 1/10 of the typical length and time scales)

Problem B 4.3. From the weather chart in today's edition of your newspaper, identify the horizontal extent of a major atmospheric feature and find the forecast wind speed. From these numbers, estimate the Rossby number of the weather pattern. What can you conclude about the importance of the Coriolis force? (Hint: When converting latitude and longitude differences in kilometres, use the eaarth's mean radius, 6371 km. Cushman-Roisin and Beckers: Physical problems 4-1)

Chapter 7.

Problem B 7.1. Show that geostrophic flow on the f-plane with flat bottom is divergence free. What are the implications for geostrophic flow if it flows over irregular bottom (Omstedt, 2005)?

Problem B 7.2. In Utopia, a narrow 200 m deep channel empties in a broad bay of varying bottom topography (Figure 7-13). Trace the path to the sea and the velocity profile of the channel outflow. Take $f = 10^{-4}\ s^{-1}$. (Solve only for straight streches of the flow and not for corners, Cushman-Roisin and Beckers: Physical problems 7-8).

Problem B 7.3. Examine vorticity dynamics by assuming that the the outflow from the Baltic Sea into the Kattegat conserves potential vorticity. What will happen with the flow when the outflow enters the much deeper Skagerrak? Illustrate how the different components in the relative vorticity may change (Omstedt, 2005)?

Chapter 8.

Problem B 8.1. Name and illustrate by figure some naturally occurring flows that may lead to convergence and divergens zones in the surface layer (Omstedt, 2005).

Problem B 8. 2. Approximate the variation of the Coriolis parameter with latitude by writing $f = f_0 + \beta_0 y$, where y is the northward coordinate (beta-plane approximation), show that the vertical velocity below the surface Ekman layer of the ocean is given by

$$\bar{w} = \frac{1}{\rho_0}\left(\frac{\partial}{\partial x}\left(\frac{\tau^y}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau^x}{f}\right)\right) - \frac{\beta_0}{f}\int_z^0 \bar{v}dz$$

Where τ^x and τ^y are the zonal and meridional wind-stress components, respectively, and $\bar{v}$ is the meriodional elocity in the geostrophic interior below the Ekman layer (Cushman-Roisin and Beckers: Physical problems 8-7).

Chapter 9.

Problem B 9.1. Prove that Kelvin waves propagate with the coast on their left in the Southern Hemisphere (Cushman-Roisin and Beckers: Physical problems 9-1).

Problem B 9.2. The Yellow Sea between China and Korea (mean latitude: 37^o N) has an average depth of 50 m and a coastal perimeter of 2600 km. How long does it take for a Kelvin way to go around the shores of Yellow Seas (Cushman-Roisin and Beckers: Physical problems 9-1)?

Problem B 9.3. Find the frequency ω of a Kelvin wave number k (Section 9-2). Is the Kelvin wave dispersive (Cushman-Roisin and Beckers: Physical problems B-5)?

Chapter 10.

Problem B 10.1. The atmospheric jet stream is a wandering zonal flow of the upper troposphere, which, by and large, determines our weather. If we ignore the variations in air density, we can model the average jet stream as a purely zonal flow, independent of height and varying meridionally according to

$$\bar{u}(y) = U\exp(-\frac{y^2}{2L^2})$$

Where the constants U and L, characteristics of the speed and width, respectively, are taken as 40 m/s and 570 km. The jet center (y = 0) is at 45^o N where $\beta_0 = 1.6110^{-11}$ $m^{-1}s^{-1}$. Is the jet stream unstable to shear waves (Cushman-Roisin and Beckers: Physical problems 10-4)?

Chapter 8old in (Cushman-Roisin,1994, will appear in chapter 20) **.**

Problem B 8old-1. Given that the North Pacific Ocean is about twice as wide as the North Atlantic Ocean and that both basins are subjected to equally strong winds, compare their boundary-layer widths and boundary current speeds (Cushman-Roisin and Beckers: Physical problems 20-2).

Problem B8old-2. Show that, as in the Northern Hemisphere, the boundary currents in ocean basins of the Southern Hemisphere are along the western boundaries. Which way are they directed (Cushman-Roisin, 1994 8-5) (Hint: Start from Ekman transport, Sverdrup transport and conservation of potential vorticity)?

Chapter 11.

Problem B11.1. The central Baltic Sea is characterized by a 60 meter deep surface layer with a salinity around 7. Below the halocline the salinity is about 10. Using the value 8 10^{-4} for the coefficient of salinity expansion, calculate the stratification frequency. What is the horizontal length scale at which rotation and stratification play comparable roles? (Hint; use the equation of state $\rho = \rho_0(1+\alpha_s S)$, Omstedt, 2005).

Problem B11.2. What terms need to be considered and neglected in the governing equations if:

a) The temporal Rossby number is much less than 1?
b) The Rossby number is much less than 1?
c) The Ekman number is much less than 1?
d) The Froude number less than 1?

Give also examples on processes in the ocean where the approximations a) to d) can be used (Omstedt, 2006).

Chapter 13.

Problem B 13.1. In a coastal ocean, the water density varies from 1028 kg/m^3 at the surface to 1030 kg/m^3 at a depth 100 m. What is the maximum internal-wave frequency? What is the corresponding period (Cushman-Roisin and Beckers: Physical problems 13-1)?

Problem B 13.2. Derive the dispersion relation for interal waves with rotation in an ocean with a coast. Are the frequency limited by f and N? (Hint: Use the Equation X in A13.1 and assume a coastal trapped wave solution according to $w = w_0 e^{i(my+nz-\omega t)-lx}$, Omstedt, 2005).

Chapter 14.

Problem B 14.1. Formulate the Richardsson number for a stratified shear flow with uniform stratification frequency N and a linear velocity profile, varying from zero at the bottom to U at a height H. Then, relate the Richardsson number to the Froude number and show that instabilities can occur only if the Froude number exceeds the value 2 (Cushman-Roisin and Beckers: Physical problems 14-3).

Problem B 14.2. In an oceanic region far away from coasts and strong currents, the upper water column is stable stratified (N = 0.015 s^{-1}). A storm passes by and, during 10 hours, exerts an averaged stress of 0.2 Nm $^{-2}$. What is the depth of the mixed layer by the end of the

storm? (For sea water take $\rho_o = 1028$ kgm^{-3}, Cushman-Roisin and Beckers: Physical problems 14-4). (Hint: Use equation 11-32 in Cushman-Roisin, 1994:
$h = (\frac{12 m u_*^3}{N^2} t)^{1/3}, m = 1.25, u_* = \sqrt{\frac{\tau}{\rho_0}}$).

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